The eqn. of an ellipse in polar coordinate

$$\frac{1}{r} = \frac{1+\varepsilon\cos\phi}{a(1-\varepsilon^2)} \qquad \text{where } a = \text{semimajor axis}$$
$$\varepsilon = \text{eccentricity}$$

Taking logarithm on each side

$$\log\left(\frac{1}{r}\right) = \log\left[\frac{1+\varepsilon\cos\phi}{a(1-\varepsilon^2)}\right]$$

Differentiating with respect to $\phi$

So, $$(-)\frac{1}{r}\cdot\frac{dr}{d\phi} = \frac{-\varepsilon\sin\phi}{1+\varepsilon\cos\phi}$$

Substituting this value in equation no. (1)

$$p_r\,dr = \frac{\varepsilon^2\sin^2\phi}{(1+\varepsilon\cos\phi)^2}\cdot p_\phi\,d\phi$$

$$\oint p_r\,dr = n_r h$$

$$p_\phi\int_0^{2\pi}\frac{\varepsilon^2\sin^2\phi}{(1+\varepsilon\cos\phi)^2}\cdot d\phi = n_r h \rightarrow (2)$$

The integral $I = \int_0^{2\pi}\frac{\varepsilon^2\sin^2\phi}{(1+\varepsilon\cos\phi)^2}\,d\phi$ can be done by integration by parts

$$\int u\,dv = uv - \int v\,du$$

let $u = \varepsilon\sin\phi$, $du = \varepsilon\cos\phi\,d\phi$

$$dv = \frac{\varepsilon\sin\phi}{(1+\varepsilon\cos\phi)^2}\,d\phi;\quad v = \frac{1}{1+\varepsilon\cos\phi}$$

$$\therefore I = \left[\frac{\varepsilon\sin\phi}{1+\varepsilon\cos\phi}\right]_0^{2\pi} - \int_0^{2\pi}\frac{\varepsilon\cos\phi}{1+\varepsilon\cos\phi}\cdot d\phi$$

$$I = -\int_0^{2\pi}\frac{\varepsilon\cos\phi}{1+\varepsilon\cos\phi}\cdot d\phi = \int_0^{2\pi}\left(\frac{1}{1+\varepsilon\cos\phi} - 1\right)d\phi$$

$= \cancel{2\pi}$